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DISTRIBUTED ORDER DIFFERENTIAL EQUATIONS MODELLING DIELECTRIC INDUCTION AND DIFFUSION

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Abstract

Distributed order differential equations (Caputo 1995, Bagley and Torvik 2000a, 2000b) are introduced in the constitutive equations of dielectric media and in the diffusion equation. To model induction phenomena the distributed order differential operators act on the induction and on the applied electric field, to model diffusion phenomena the distributed order differential operators acts on the flux and on the pressure gradient. The solution of the classic problems of dielectrics and of diffusion are first found in the frequency domain, with a discussion of their filtering properties, and then obtained, with closed form formulae, also in the time domain. For the dielectric media the Green function is decreasing asymptotically to zero and acts on the input as a low pass filter. The Green function is found for the classic problem of the diffusion, it acts also as a low pass filter. The major difference with the case when a single fractional derivative is present in the constitutive equations of dielectric media, and also in the diffusion equation, is that the solutions found here and the associated filters are potentially more flexible to represent more complex systems (Caputo 1995).

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Key Words and Phrases: distributed order, fractional order, differential equations, constitutive equations, dielectric media, diffusion

1 Introduction

The use of fractional order derivatives to model physical phenomena has been diffused in the last few decades in basic mathematics (Bagley and Torvik 2000a and 2000b, Caputo 1994 and 1995, Flores and Osler 1999, Kiryakova 1994, Luchko and Gorenflo 1999, Oldham and Spanier 1974, Podlubny 1994a, 1994b and 1999, Ross 1975) and in applied research (Bagley 1991, Belfiore and Caputo 2000, Caputo 2000, Caputo and Plastino 1998 and 1999, Jacquelin 1984 and 1991, Körnig and Müller 1989, Le Mehaute and Crépy 1983, Mainardi 1993, Wyss 1986).

Fractional differential equations consisting of sums of fractional order derivatives have been extensively studied by Podlubny (1994a, 1994b, 1999). Fractional differential equations where the order of differentiation has been integrated over a given range, and therefore there is no single order of differentiation, have been introduced and studied (Caputo 1967, 1995) in order to represent phenomena where the order of differentiation varies in a given range. Bagley and Torvik (2000a and 2000b) studied extensively these distributed order differential equations (DODE) and developed a very interesting mathematical theory.

In this note we shall use DODE in modelling the phenomena of anelasticity and dielectric induction and of diffusion, which have been already considered in the literature with constitutive equations containing fractional derivatives of a given order (Bagley 1991, Mainardi 1996, Schneider and Wyss 1989, Caputo 2000), assuming now that the fractional order is integrated over a given range.

Concerning the dielectric induction we will find the response to a generic perturbation for the diffusion, a half space is modelled where the boundary values are applied to the plane limiting it, the problem solved is the computation of the Green function of the pressure in the cases when a pressure, constant in time, is applied on all the surface of the boundary plane, while the half space is initially at zero pressure.

All problem are solved with the use of the Laplace Transform (LT), closed form formulae are obtained in the time domain for the response to perturbation in dielectrics and in the diffusion of pressure.

2 The DODE time domain solution for dielectric media

The interpretation of the observed broadening of the spectral lines in all physical phenomena is generally made assuming an exponential decay of the modes, often without the support of a physical model, sometime on the generic assumption of non linearity as the cause of the phenomenon.

The almost frequency independent specific dissipation (constant Q model), which is observed in the decay of the free modes of anelastic materials, led to models of the stress-strain relations, represented with the introduction of time in the relations, which imply the observed Q. The same models should naturally represent also the dispersion of the waves.

The most succesfully used stress-strain relations in the representation of the dispersion and of the dissipation in anelastic media are those which contain fractional order derivatives (FOD), as shown by Bagley (1991) and Körnig and Müller (1989). In the frequency domain they lead to relations, between stress and strain, and to indexes of refraction in which the imaginary frequency appears elevated to a rational power.

In the practical works with dielectrics the most used representation of the frequency dependent dielectric parameter is that of Cole and Cole (1941) in which the relation between the applied electric field and the induction contains the imaginary frequency elevated to a rational power. Caputo and Mainardi (1971) showed that this type of constitutive equation in the frequency domain implies, in the time domain, a relation containing a fractional order differentiation. Therefore, also for dielectric media, in the time domain, the relation between the applied electric field and the induction is expressed by means of FOD.

Concerning the physical implications of the introduction of the FOD, we may qualitatively consider this type of derivative as representative of the memory of the medium relative to its precedent physical conditions. From a quantitative and rigorous point of view the derivative of rational order implies a multivalued index of refraction and therefore (Caputo 1996) a monochromatic electromagnetic wave, incident the medium with memory with direction normal to its surface, or originated by a source in the medium itself, will split into a set of waves with the same frequency, but different wavelengths and velocities, which interfere, have beats and tunnels. It has been shown by Cisotti (1911) that the memory introduced in the dielectric constant implies that the phenomenology of the medium is irreversible. It has been shown also that, in anelastic media, the stress strain relations with FOD are causal (Bagley (1991).

In an extensive study of energy storage in electric networks Jacquelin (1984, 1991) used the complex frequency dependent impedance represented by inserting FOD in the relation between the induced electric current and the electric field. Jacquelin (1984, 1991) discussion is practically extended to almost all possible circuits; he illustrates his results in the frequency domain at steady state, when the Fresnell terms of the fractional order derivatives are nil, and his technique, based on the observations at many frequencies, relies on convolution for the retrieval of the response in the time domain.

In general we may state that the formalism of FOD is now spread to many linear approaches of the studies of dissipative and dispersive properties of many anelastic and dielectric media (e.g. Caputo 1967, 1976, 1996, Caputo and Plastino 1998, Bagley 1991, Jacquelin 1984, 1991, Körnig and Müller 1989, Le Mehaute and Crépy 1983, Mainardi 1993, Pelton et al. 1983).

In this note we will consider an extension, of the constitutive relations in which FOD are present, to the case when the memory mechanism operating in the medium is represented by FOD whose order covers a continuum in a given range. Expressing this concept with formulae we begin with the basic constitutive equation of elastic and dielectric media in one dimension which is generally written

$$d(t) = \varepsilon\ e(t) \tag{1}$$

where, in the case of anelastic media, *d(t)* is the stress and *e(t)* the strain, ε is the elastic parameter, in the case of dielectric media, *d(t)* is the induction, *e(t)* is the electric field and ε is the dielectric constant. We will devote here our attention to the dielectric media, the extension of the following considerations to the anelastic media is almost obvious.

In this note we will assume that (1) be in fact

$$(1/(b-a)) \int_a^b \alpha(z) e^{(z)}(t) dz + \beta e(t) = (1/(f-c)) \int_c^f \gamma(z) d^{(z)}(t) dz + \delta d(t) \tag{2}$$

where $0 \leq c < f < 1$, $0 \leq a < b < 1$,

$$f^{(z)}(t) = \partial^z f(t)/\partial t^z = (1/\Gamma(1-z)) \int_0^t f'(u) du/(t-u)^z \tag{3}$$

$\alpha(z)$ and $\gamma(z)$ have dimension of s^z, β and δ are dimensionless and $0 < z < 1$. The division by the length of the interval of integration will normalise the operation and ensures that, when $b \to a$, the integration will lead to the simple FOD of order a . Taking the LT of both sides we find

$$E(s) \left[(1/(b-a)) \int_a^b s^z \alpha(z) dz + \beta \right] = D(s) \left[(1/(f-c)) \int_c^f s^z \gamma(z) dz + \eta \right] \tag{4}$$

where s is the LT variable, $E(s)$ and $D(s)$ are the LT of $d(t)$ and $e(t)$ respectively and it is assumed that $e(0) = d(0) = 0$. To go from (2) to (3) we used the property of the interchange between the integrals with respect to z and that of the LT and also the property concerning the LT of a FOD which is here specialised to the case when $n = 0$ (Caputo 1967)

$$LT(\partial^z f/\partial t^z) = s^z F(s) - s^{z-1} f(0) \tag{5}$$

We will discuss here the case of interest, for the possible applications when solving inverse problems, when $\alpha(z) = \alpha, \gamma(z) = \gamma$ with γ and α constant, (2), in this case, is

$$D = E \left[(\alpha/(b-a))(s^b - s^a)/log(s) + \beta \right] / \left[(\gamma/(f-c))(s^f - s^c)/log(s) + \eta \right] \tag{6}$$

$$D(s) = E(s)\beta\Omega(s,a,b)\Psi(s,c,f)/\eta = E(s)\Lambda(s)$$

$$\Lambda(s) = \left[(\alpha/(b-a))(s^b - s^a)/log(s) + \beta\right] / \left[(\gamma/(f-c))(s^f - s^c)/log(s) + \eta\right]$$

where

$$\Psi(s,c,f) = 1/\left\{(\gamma/(f-c)\eta)(s^f - s^c)/ln(s) + 1\right\} \tag{7}$$

$$\Omega(s,a,b) = (\alpha/(b-a)\eta)(s^b - s^a)/ln(s) + 1$$

$$d = (\eta/\beta)\left\{LT^{-1}\left[\Psi(s,c,f)\right]\right\} \star \left\{LT^{-1}\left[\Omega(s,a,b)/s\right]\right\} \star e' \tag{8}$$

Which is the formal solution of the direct problems in the LT domain, of the computation of *d(t)* when *e(t)* is given. The real and imaginary parts of the dielectric constant is illustrated in figures 1 and 2. In figures 3 and 4 is shown the difference between the case considered here of distributed order fractional derivatives and of single order fractional derivative; note that the order of the single order derivative is median of the interval of distributed order derivatives.

The filtering properties of the factor of E(s) in equation (6) are expressed by

$$|\Lambda(\omega)| = ((f-c)/(b-a))$$

$$\left|\alpha^2(\omega^{2b} - \omega^{2a}) + (b-a)^2\beta^2 - 2\alpha^2\omega^{a+b}\cos(b-a)\pi/2 - 2\alpha\beta(b-a)\right.$$

$$\tag{8'}$$

$$\left.(\omega^b\cos(b)\pi/2 - \omega^a cos(a)\pi/2\right|^{1/2} / \left|\gamma^2(\omega^{2f} - \omega^{2c}) + (f-c)^2\eta^2 - 2\gamma^2\omega^{f+c}\right.$$

$$\left.\cos(f-c)\pi/2 - 2\gamma\eta(f-c)(\omega^f\cos(f)\pi/2 - \omega^c cos(c)\pi/2\right|^{1/2}$$

and it is verified that

$$\lim_{\omega\to\infty}\Lambda(\omega) = 0$$

when $b < f$

$$\lim_{\omega\to\infty}\Lambda(\omega) = \infty$$

when $b > f$

$$\lim_{\omega\to\infty}\Lambda(\omega) = \alpha(f-c)/\gamma(f-a)$$

when $b = f$

$$\lim_{\omega\to 0}\Lambda(\omega) = \beta/\eta$$

which imply the possibility of low and high pass filtering as shown in figure 5.

In the simpler case when $\gamma = 0$ the function $\psi(t,a,b) = LT^{-1}\Psi(s,a,b)$ is given in equations (A1) and (A2) of the appendix A, we obtain

$$e(t) = (\eta/\beta)d(t) \star \psi(t,a,b) =$$

$$= (\beta/\eta\pi)d(t) \star \int_0^\infty exp(-rt)\left\{\left[Ar^b\sin(b\pi) - Ar^a\sin(a\pi) + \pi\right]log(r) + \right.$$

$$\left. -\pi\left[Ar^b\cos(b\pi) - Ar^a\cos(a\pi) + log(r)\right]\right\}dr/ \tag{9}$$

$$\left\{\left[Ar^b\cos(b\pi) - Ar^a\cos(a\pi) + log(r)\right]^2 + \left[Ar^b\sin(b\pi) - Ar^a\sin(a\pi) + \pi\right]^2\right\}$$

where $A = \alpha/\beta(b-a)$.

The function $\psi(t,a,b)$ is monotonically decreasing with increasing t, moreover $\psi(\infty,a,b) = 0^{1-b}$and $\psi(0,a,b) = \infty^{1-a}$. The specific dissipation $Q^{-1}(\omega) = Im\Psi(i\omega,a,b)/Re\Psi(i\omega,a,b)$ has the values $Q^{-1}(0) = 0$ and $Q^{-1}(\infty) = tan(b\pi/a)$. The filtering acting on *d(t)* is represented by $\Psi(s,a,b)$ and is a low pass filter with values $\Psi(0,a,b) = 1$ and $\Psi(\infty,a,b) = 0$.

The direct problem, when *E(s)* is given and *D(s)* is computed from (5), has an apparent complication since it is readily seen that the LT of the response function is not converging for $s \to \infty$. However we may go around this with a little artifice, we write the equation (5) as follows dividing and multiplying by s

$$D(s) = (\beta/\eta)sE\left[\alpha K(s,a,b)/\beta(b-a) + 1/s\right],$$

$$\Omega(s,a,b) - 1 = K(s,a,b) = (s^b - s^a)/slog(s) \tag{10}$$

$\kappa(t,a,b) = LT^{-1}K(s,a,b)$ is discussed in appendix A (equation (A2')) which, since *e(0) = 0*, gives

$$d(t) = (\beta/\eta)\left\{e(t) + e'(t) \star (\alpha/\beta(b-a))\kappa(t,a,b)\right\} =$$

$$= (\beta/\eta)\left\{e(t) + e'(t) \star (\alpha/(\beta(b-a)\pi)\int_0^\infty exp(-rt)\left[r^{b-1}(log(r\sin(b\pi)) - \right.\right.$$

$$\left.\left. +\pi\cos(b\pi)) - r^{a-1}(log(r\sin(a\pi)) - \pi\cos(a\pi))\right]dr/\left[(log(r))^2 + \pi^2\right]\right\} \tag{11}$$

or

$$d(t) = (\beta/\eta)e'(t) \star \left\{1 + (\alpha/\beta(b-a)\pi)\int_0^\infty exp(-rt)\left[r^{b-1}(log(r\sin(b\pi)) - \right.\right.$$

$$+\pi\cos(b\pi)) - r^{a-1}(log(r\sin(a\pi)) - \pi\cos(a\pi))\right] dr/\left[(log(r))^2 + \pi^2\right]\right\}$$

The function $\kappa(t,a,b)$ is a monotonically decreasing function of t, as discussed in appendix A where we find $\kappa(\infty,a,b) = 0^a$ and $\kappa(0,a,b) = \infty^b$.

The filter *K(s,a,b)* acting on $e'(t)$ is a low pass filter with values $K(0,a,b) = \infty$, $K(\infty,a,b) = 0$.

The explicit time domain form of (6) is obtained with the combined use of both functions $\kappa(t,a,b)$ and $\psi(t,c,f)$ we find

$$d(t) = (\beta/\eta)\left\{e(t) + e'(t) \star (\alpha/(\beta(b-a))\kappa(t,a,b)\right\} \star \{\psi(t,c,f)\} =$$

$$= (\beta/\eta)\left\{e(t) + e'(t) \star (\alpha/\beta(b-a)\pi)\int_0^\infty exp(-rt)\left[r^{b-1}(log(r\sin(b\pi)) -\right.\right.$$

$$+\pi\cos(b\pi)) - r^{a-1}(log(r\sin(a\pi)) - \pi\cos(a\pi)\right] dr/\left[(log(r))^2 + \pi^2\right]\right\} \star \quad (12)$$

$$\int_0^\infty exp(-rt)\left\{\left[Br^f\sin(f\pi) - Br^c\sin(c\pi) + \pi\right] log(r) - \pi\left[Br^f\cos(f\pi) -\right.\right.$$

$$+Br^c\cos(c\pi) + log(r)\right]\right\} dr/\left\{\left[Br^f\cos(f\pi) - Br^c\cos(c\pi) + log(r)\right]^2 +\right.$$

$$\left. + \left[Br^f\sin(f\pi) - Br^c\sin(c\pi) + \pi\right]^2\right\}$$

where $B = \gamma/\eta(f-c)$.

Concerning the filtering properties of the general case α, β, γ, $\eta \neq 0$ we consider the particular case when $b = f$. It is seen that when $\beta/\eta < (\alpha/\gamma)(f-c)/(f-a)$ the filtering is a high pass. When $\beta/\eta > (\alpha/\gamma)(f-c)/(f-a)$ the filtering is a low pass. It is also seen that when the percent of *e(t)* subject to memory is larger than that in *d(t)* (that is $\alpha/\beta(f-a) > \gamma/\eta(f-c)$) then the memory is a high pass filter ; when instead the percent of memory in *d(t)* is larger than that in *e(t)* (that is $\alpha/\beta(f-a) < \gamma/\eta(f-c)$) then the memory is a low pass filter as is illustrated in figure 5.

3 The diffusion

The studies of diffusion in porous media are very important in various fields of science and also for social needs especially in the case of water and pollutants. The diffusion is presently modelled to represent the diversified forms of deviations from the classic constitutive law and several complex mathematical methods for modelling the phenomenon have been introduced, ranging from non linearity to statistical mechanics and memory formalisms.

Concerning the diffusion we will consider one-dimensional problems in a half space bounded by a plane where we assume the origin of the x axis oriented positive towards the medium and the classic constitutive equation of Fick or Darcy binding the flux *q(x,t)* to the gradient of the pressure *u(x,t)*

$$q(x,t) = -\ p\ \rho_0\ grad(u(x,t)) \tag{13}$$

where ρ_0 is the density of the fluid in its undisturbed condition and p the ratio of the permeability of the medium to the fluid viscosity.

Christakos et al. (1995) used stochastic diagrammatic analysis of groundwater flow in heterogeneous porous media, Barry and Sposito (1989) found an analytical solution of a convection dispersion model with time dependent transport coefficient, Hu and Cushman (1991) used a non equilibrium statistical mechanical derivation of a non-local Darcy's law for unsaturated/saturated flow.

Here we will generalise the constitutive equation (13) then write as follows

$$(1/(f-c))\int_c^f [\gamma(\partial^z/\partial t^z)\overline{q}(x,t)]\,dz + \eta\overline{q}(x,t) =$$

$$= -(1/(b-a))\int_a^b [\alpha(\partial^z/\partial t^z)grad(u(x,t))]\,dz - \beta grad(u(x,t)) \tag{14}$$

where η, γ, α and β are parameters with the appropriate dimensions obviously different than in equation (2) ; the same letters of equation (2) are used in order to symplify the discussion since equation (2) and equation (14), from the point of view of the memory are formally the same .

Taking the LT of (14), indicating with s the LT variable and with *U(x,s)* and *Q(x,s)* the LT of the pressure and the flux respectively, we obtain

$$(1/(f-c))\int_c^f \left[\gamma s^z Q(x,s) - \gamma s^{z-1} q(x,0)\right] dz + \eta Q(x,t) =$$

$$= -(1/(b-a))\int_a^b \left[\alpha s^z grad(U(x,s)) - \alpha s^{z-1} grad(u(x,0))\right] dz - \beta grad(U(x,t)) \tag{15}$$

$$Q(s,x)\left\{(\gamma/(f-c))\int_c^f s^z dz + \gamma\right\} - (\gamma/(f-c))q(x,0)s^{-1}\int_c^f [s^z]\,dz =$$

$$= -grad(U(x,s))\left\{(\alpha/(b-a))\int_a^b s^z dz + \beta\right\} + (\alpha/(b-a))grad(u(x,0))\int_a^b s^{z-1}dz$$

Performing the integrations we finally find the LT version of the constitutive equation relating the gradient of the pressure to the flux

$$(\gamma/(f-c))\left[(s^f-s^c)/log(s)\right]Q(x.s)-(\gamma/(f-c))((s^{f-1}-s^{c-1})/$$

$$log(s))q(x,0)+\eta Q(x,s)=-(\alpha/(b-a))\left[(s^b-s^a)/log(s)\right]grad(U(x,s))+ \quad (16)$$

$$+(\alpha/(b-a))((s^{b-1}-s^{a-1})/log(s))grad(u(x,0))-\beta grad(U(x,s))$$

where, to simplify the algebra, we will assume that

$$q(x,0)=0$$

$$grad(u(x,0))=0$$

(16')

$$(or\ u(x,0)=constant)$$

The filter acting on grad U is Λ(s), the same as that acting on E(s) of equation (6), is therefore valid also for grad U the discussion made for the filtering acting on E(s).

Then we must consider the constitutive equation which binds the pressure of the fluid *u(x,t)* to the density $\rho(x,t)$ which appears in the continuity equation,

$$u=k\rho(x,t) \quad (17)$$

where k is a constant.

Equation (17) could contain a memory formalism since the fluids are elastic and some of their physical and/or chemical properties may change in time due to interactions with the medium.

A memory formalism in the constitutive equation (17) was formally considered in a previous study (Caputo 2000), in the present discussion we will neglect these interaction and assume that they are transferred to the memory inserted in the constitutive equation (14).

Finally we must also consider the continuity equation

$$\partial q(x,t)/\partial x+k\partial\rho(x,t)/\partial t=0 \quad (18)$$

which is associated to the time derivative form of equation (17)

$$\partial u(x,t)/\partial t=k\partial\rho(x,t)/\partial t \quad (19)$$

to give

$$\partial q(x,t)/\partial x = -\partial u(x,t)/k\partial t \tag{20}$$

or taking the LT of both member

$$\partial Q(x,t)/\partial x = -sU(x,s)/k + u(x,0)/k \tag{20'}$$

Comparing equations (20') and (16) we finally find

$$\partial Q(x,s)/\partial x =$$

$$= -\left\{(\alpha/(b-a))\left[(s^b - s^a)/log(s)\right] + \beta\right\} / \left\{(\gamma/(f-c))\left[(s^f - s^c)/log(s)\right] + \eta\right\}$$

$$\partial^2 U(x,s)/\partial x^2 + s^{-1}\left[(\gamma/(f-c))((s^f - s^c)/log(s))\partial q(x,0)/\partial x - (\alpha/(b-a))\right.$$

$$\left.((s^{b-1} - s^{a-1})/log(s))\partial^2 u(x,0)/\partial x^2\right] / ((1/(f-c))$$

$$\left[\gamma(s^f - s^c)/log(s) + \eta\right])$$

or, considering the conditions (16'),

$$\partial Q(x,s)/\partial x =$$

$$= -\left[\left\{(\alpha/(b-a))\left[(s^b - s^a)/log(s)\right] + \beta\right\} / \left\{(\gamma/(f-c))\left[(s^f - s^c)/log(s)\right] + \eta)\right\}\right]$$

$$\partial^2 U(x,s)/\partial x^2$$

$$\partial^2 U(x,s)/\partial x^2 =$$

$$\left[\left\{(\gamma/(f-c))\left[(s^f - s^c)/log(s)\right] + \eta\right\} / \left\{(\alpha/(b-a))\left[(s^b - s^a)/log(s)\right] + \beta\right\}\right]$$

$$\{sU(x,s)/k - u(x,0)/k\} \tag{21}$$

We should now ask ourselves how to select the values of the parameters η, γ, α, β for the applications. To this purpose we should discuss equation (21) and first distinguish between slow and fast diffusion. To this aim it is worthwhile to consider the following two cases 1) $\beta = \gamma = 0$ and 2) $\alpha = \eta = 0$.

In case 1) the memory acts on the flux and gives to the previous values of the flux weights which are decreasing as is increasing the time separation from the beginning of the flux. In case 2) the memory acts on the pressure gradient and gives to the previous values of the pressure gradients weights which are decreasing as is increasing the time separation from the beginning of the pressure gradient.

In case 1) ($\beta = \gamma = 0$) equation (21) is

$$\partial^2 U(x,s)/\partial x^2 =$$

$$= \{sU(x,s)/k - u(x,s)/k\} \left\{\eta/(\alpha/(b-a)) \left[(s^b - s^a)/log(s)\right]\right\} \qquad (22)$$

Taking the limit of equation (22) for $b \to a$, since $0 < a < b < 1$ and $0 < 1 - b < 1$ it is seen that equation (22) reduces to the case of slow diffusion and therefore for analogy we will say that case 1), when the memory acts on the flux, is of slow diffusion since the integration is taken over a range of z contained in the range [0,1].

In the case 2) ($\alpha = \eta = 0$) equation (21) is

$$\partial^2 U(x,s)/\partial x^2 = \left\{(\gamma/(f-c)) \left[(s^f - s^c)/log(s)\right]/\beta\right\} \{sU(x,s)/k - u(x,0)/k\} \qquad (23)$$

Taking the limit of (23) for $f \to c$ it is seen that, since $0 < c < f < 1$ and $2 > 1 + f > 1$, equation (23) takes the form of that of fast diffusion. We will therefore say that in case 2), when the memory acts on the pressure gradient, the equation (21) represents a fast diffusion which we will consider in the following.

More generally considering the equation (21) we note that for large values of s, in the case 1), taking the limit for $f \to c$ we obtain the case studied by Caputo (1999) where the velocity of propagation of the maximum pressure caused by a pulse of pressure at the boundary is $x^{-[1-(b-f)]/[1+(b+f)]}$; we now consider the constitutive equation (16) for sufficiently large values of s which, with the conditions (16'), is

$$Q(x,s) = -((f-c)\alpha/(b-a)\varepsilon)s^{b-f} grad(U(x,s)) \qquad (23')$$

and therefore when $f < b$ the flux results from a high pass filtering of the gradient pressure and the velocity of propagation of the peaks of the pressure decreases less rapidly with increasing x than in the case when $f > b$ and the flux results from a low pass filtering of the pressure gradient. In other words we find that, in general, the slow diffusion is associated to a high pass filtering of the pressure gradient while the fast diffusion is associated to a low pass filtering.

The reference diffusion is therefore for $f = b$ which separates the slow from the fast diffusion.

Concerning the filtering properties associated to equation of the constitutive equation (16), with the conditions (16'), in the general case we see that when $f > b$ it is a high pass filtering while when $f > b$it is a low pass filtering. When $b = f$ and $\alpha(f-c)/\gamma(f-a) > \beta/\eta$ (that is when the percent of memory acting on *grad(p(x,t))* is larger than that acting on *q(x,t)* it is a high pass filtering while when $\alpha(f-c)/\gamma(f-a) < \beta/\eta$ (that is when the percent of memory acting on *grad(p(x,t))* is smaller than that acting on *q(x,t)* it is a low pass filtering as is illustrated in figure 5.

We will now consider the signalling problem in the case when $u(0,t) = P$, $u(x,0) = 0$ noting that they satisfy the condition (16').

The formal solution of equation (21) is

$$U(x,s) =$$

$$= A(s)exp\left[x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s) + \eta s)/k(\alpha(s^b - s^a)/(b-a)\right.\right.$$

$$\left.\left.log(s) + \beta)\right)^{1/2}\right] + B(s)exp\left[-x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s) + \eta s)/\right.\right.$$

$$\left.\left.k(\alpha(s^b - s^a)/(b-a)log(s) + \beta)\right)^{1/2}\right] \tag{24}$$

The LT domain solution converging at infinity in the half space and satisfying the boundary and initial condition, that is when $u(0,t) = P$, $u(x,0) = 0$ (a signalling problem), is then

$$U(x,s) =$$

$$= (P/s)exp\left[-x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s) + \eta s)/k(\alpha(s^b - s^a)/\right.\right.$$

$$\left.\left.(b-a)log(s) + \beta)\right)^{1/2}\right] \tag{25}$$

The solution in the time domain is therefore

$$u(x,t) = P \star \sigma(x,t), \quad \sigma(x,t) = LT^{-1} \sum(x,s)$$

$$\sum(x,s) = exp\left[-x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s) + \eta s)/k(\alpha(s^b - s^a)/\right.\right.$$

$$\left.\left.(b-a)log(s) + \beta)\right)^{1/2}\right] \tag{26}$$

The possibility of the existence of the LT^{-1} of (26) is ensured since $1+f > b$. With the EVT it is verified that $\sigma(x,0) = 0$, $\sigma(x,\infty) = 0$ and that the initial and boundary conditions for *u(x,t)* are also satisfied.

In the following we will consider the simpler signalling case $\alpha = 0$, $\beta = 1$, $\eta = 0$ of fast diffusion, we find from (25)

$$U(x,s) = -(P/s)exp\left[-x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s))/k\right)^{1/2}\right] \qquad (27)$$

Introducing $\sigma(x,t)$ found in equations (A3) through and (A10) of the Appendix A we obtain

$$u(x,t) = (1/\pi)P \star \int_0^\infty exp(-rt)$$

$$\left\{exp\left[W(U^2+V^2)^{1/4}cos((0.5)\tan^{-1}(V/U))\right]\right\}$$

$$sin((0.5)tan^{-1}(V/U))dr \qquad (28)$$

where

$$l = r^f\cos(f\pi) - r^c\cos(c\pi), \quad m = r^f\sin(f\pi) - r^c\sin(c\pi), \quad C = (\gamma/(f-c))^{1/2} \qquad (29)$$

$$W = -xC(r/((log(r))^2+\pi^2)^{1/2}), \quad U = -l\ log(r) - m\pi, \quad V = -m\ log(r) + l\pi$$

Performing the convolution we find the final solution

$$u(x,t) = (P/\pi)\int_0^\infty [(1-exp(-rt))/r]$$

$$\left\{exp\left[W(U^2+V^2)^{1/4}cos((0.5)\tan^{-1}(V/U))\right]\right\}$$

$$sin((0.5)tan^{-1}(V/U))dr \qquad (30)$$

The filter $\sum(x,s)$ acting on the boundary value P of (29) is a low pass filter, it has values $\sum(x,\infty) = 0$, $\sum(x,0) = exp(-x(\gamma/k\beta)^{1/2})$.

$\sigma(x,t)$ may be considered the Green function of the problem discussed, that is when $u(0,t) = k$, $u(x,0) = 0$.

In similar manner is found the solution of the other classic problem when *u(x,0)* $= P =$ constant, $u(0,t) = 0$, (Cauchy problem), we find

$$U(x,s) = (P/s)\left\{1 - exp\left[-x\left((\gamma(s^{f+1} - s^{c+1})/(f-c)log(s))/k\right)^{1/2}\right]\right\}$$

whose inverse LT is readily obtained from equation (28).

4 The flux

We see in this paragraph that the equation governing the diffusion written with the pressure as unknown is the same as that write with the flux as unknown. In fact eliminating *U(x,t)* between equations (16) and (21) we find

$$\partial^2 Q(x,s)/\partial t^2 =$$

$$= [sQ(x,s)/k]\left[(\gamma/(f-c))(s^f - s^c)/log(s) + \eta\right] / \left[(\alpha/(b-a))(s^b - s^a)/\right.$$

$$\left. log(s) + \beta\right] + (\partial u(x,0)/\partial x)/k$$

The same property is valid also for a generic constitutive equation of the type

$$F_2(s)\partial U(x,s)/\partial x = -Q(x,s)F_1(s)$$

However we note that the solutions found for the pressure are associated to the boundary and initial problems of the pressure and therefore they are valid only for the corresponding problem of the flux.

If we want the flux caused by the pressure distribution of the signalling problem of pressure, for instance, we must substitute the LT of the pressure gradient in constitutive equation (16) and find its solution with the LT inversion.

The solutions is obtained isolating *Q(x,s)* is equation (16), finding the inverse LT of the factor of *grad U(x,s)* and use the convolution theorem since *grad u(x,t)* is known with the same boundary and initial conditions.

Appendix A. Computation of some LT^{-1}used in the present note.

We shall perform here the computation of some LT^{-1} appearing in the previous sections and begin with $\psi(t) = LT^{-1}\Psi(s,f,c)$ of formula (7)

$$\psi(t) = LT^{-1}\Psi(s,f,c) = LT^{-1}\left[1/\left\{\gamma/\beta(f-c))(s^f - s^c)/ln(s) + 1\right\}\right] \tag{A1}$$

It may seem that the function $\Psi(s,c,f)$ in (B1) has a pole in $s = 1$, however it is readily verified that $\lim_{p\to 1}\Psi(s,c,f) = 1/\left\{(\gamma/\beta(f-c)) + 1\right\}$. It is also verified that $\lim_{p\to 0}\Psi(s,c,f) = 1$. The$LT^{-1}\Psi(s.c.f)$ is obtained integrating $LT^{-1}\Psi(s,c,f)exp(st)$ along the path of figure 6 extending the outer radius to infinity and the inner one to zero. Setting $s = r\,exp(i\theta)$ with r and θ polar coordinates in the complex plane, we find that, when $r \to \infty$, the contributions from all parts of the circuit are nil except those on the negative real axis and on the straight line in the positive real plane and obtain (e.g. Caputo 1994)

$$\psi(t) = (1/2i\pi)\int_{\infty}^{0} exp(-rt)dr/\left[1 + A\left\{r^f exp(if\pi) - r^c(ic\pi)\right\}/(log(r) + i\pi)\right] +$$

$$-(1/2\pi)\int_{0}^{\infty} exp(-rt)dr/\left[1 + A\left\{r^f exp(-if\pi) - r^c(-ic\pi)\right\}/(log(r) - i\pi)\right] =$$

$$= (1/2i\pi)\int_{0}^{\infty} exp(-rt)dr\left\{1/\left[1 + A\left\{r^f exp(if\pi) - r^c(ic\pi)\right\}/(log(r) + i\pi)\right] + \right.$$

$$\text{(A1')}$$

$$\left. -1/\left[1 + A\left\{r^f exp(-if\pi) - r^c(-ic\pi)\right\}/(log(r) - i\pi)\right]\right\}$$

where $A = \gamma/\beta(f - c)$. (A1') is an integral whose integrand is the difference of two ratios of complex conjugate numbers which leads to an imaginary expression this, in turn gives, after straightforward computations,

$$\psi(t) = (1/\pi)\int_{0}^{\infty} exp(-rt)\left\{\left[Ar^f \sin(f\pi) - Ar^c \sin(c\pi)\right] log(r) + \right.$$

$$\text{(A2)}$$

$$\left. -\pi\left[Ar^f \cos(f\pi) - Ar^c \cos(c\pi)\right]\right\} dr/$$

$$\left\{\left[Ar^f \cos(f\pi) - Ar^c \cos(c\pi) + log(r)\right]^2 + \left[Ar^f \sin(f\pi) - Ar^c \sin(c\pi) + \pi\right]^2\right\}$$

The expression (A2) gives $\psi(\infty) = 0$, as one may verify also with the EVT applied to $\Psi(s)$ since $1 > f > c > 0$. With the EVT we find also $\psi(0) = \infty^{1-c}$ and $\psi(\infty) = 0^{1-f}$.

Concerning the direct problem and formula (10) we integrate $K(s)exp(st)$ along the path of figure 6, which gives an integral whose integrand is the difference of two ratios of complex conjugate numbers. This difference leads to an imaginary expression of the integrand which in turn gives:

$$\kappa(t) = LT^{-1}\left[(s^b - s^a)/slog(s)\right] = (1/\pi)\int_{0}^{\infty} exp(-rt)\left[r^{b-1}(log(r\sin(b\pi)) - \right.$$

$$\text{(A2')}$$

$$\left. + \pi\cos(b\pi)) - r^{a-1}(log(r\sin(a\pi)) - \pi\cos(\alpha\pi))\right] dr/\left[(log(r))^2 + \pi^2\right]$$

(A2') gives $\kappa(\infty) = 0$, as one may verify also with the EVT applied to *K(s)* since $1 > b > a > 0$.

With the EVT we obtain also $\kappa(0) = \infty^b$, $\kappa(\infty) = 0^a$.

Concerning the diffusion of pressure we must compute the $LT^{-1}\sum(s)$appearing in (26), which we report here

$$\sigma(t) = LT^{-1}\sum(s)$$

$$\sum(s) = exp\left[-x\left(v(s^{f+1} - s^{c+1})/(f-c)log(s)\right)^{1/2}\right] \tag{A3}$$

We first note that $\sum(0) = 1$, $\sum(1) = 1$ and integrate $\sum(s)exp(st)$ again along the path of figure 6 finding

$$\sigma(x,t) = (1/2i\pi)\left\{\int_{\infty}^{0}(exp(-rt))exp(-xC\left\{-r^{f+1}\cos(f\pi) + r^{c+1}\cos(c\pi)+\right.\right.$$

$$\left.+i\left[-r^{f+1}\sin(f\pi) + r^{c+1}\sin(c\pi)\right]\right\}^{1/2}/\left\{log(r)+i\pi\right\}^{1/2})dr+$$

$$-\int_{0}^{\infty}(exp(-rt))exp(-xC\left\{-r^{f+1}\cos(f\pi) + r^{c+1}\cos(c\pi)+\right. \tag{A4}$$

$$\left.\left.+i\left[-r^{f+1}\sin(f\pi) + r^{c+1}\sin(c\pi)\right]\right\}^{1/2}/\left\{log(r)-i\pi\right\}^{1/2})dr\right\}$$

$$C = (\gamma/(f-c))^{1/2}$$

which leads to the integral of the difference of two exponentials whose exponents are ratios of complex conjugate numbers which in turn gives an imaginary expression ; after straightforward computations, we obtain

$$u(x,t) = (1/\pi)P\star\int_{0}^{\infty} exp(-rt)$$

$$exp\left[W(U^2+V^2)^{1/4}\cos((0.5)\tan^{-1}(V/U))\right]\sin((0.5)\tan^{-1}(V/U)) \tag{A5}$$

where

$$l = r^f\cos(f\pi) - r^c\cos(c\pi), \quad m = r^f\sin(f\pi) - r^c\sin(c\pi)$$

$$W = -xC(r/((log(r))^2+\pi^2)^{1/2}), \quad U = -l\ log(r) - m\pi, \quad V = -mlog(r) + l\pi$$

It is seen that (A4) gives $\sigma(\infty) = 0$, which is verified with the EVT applied to $\sum(s)$ since $1 > f > c > 0$. With the EVT it is also seen that $\sigma(0) = 0$.

Figure captions

Figure 1. Imaginary part of the complex dielectric or anelastic parameter for several values of the parameters defining it in formula (6). In all curves is assumed $b = f$, $c = a$, $\eta/\beta = 1$. In the solid curve $\gamma/(b-a)\eta = 0.5$, $\alpha/\beta(b-a) = 1$, in the dashed curve $\gamma/(b-a)\eta = 0.05$, $\alpha/\beta(b-a) = 1$; also $b = 0.6$, $a = 0.5$. The curve with $\gamma/(b-a)\eta = 0.005$, $\alpha/\beta(b-a) = 1$, would overlap the solid one. The dotted curve is with $a = 0.2$, $b = 0.3$ and has the other parameter values as the solid curve. The abscissa is frequency.

Figure 2. Real part of the complex dielectric or anelastic parameter for several values of the parameters defining it in formula (6). In all curves is assumed $b = f$, $c = a$, $\eta/\beta = 1$. In the solid curve $\gamma/(b-a)\eta = 0.5$, $\alpha/\beta(b-a) = 1$, in the dashed curve $\gamma/(b-a)\eta = 0.05$, $\alpha/\beta(b-a) = 1$. The curve with $\gamma/(b-a)\eta = 0.005$, $\alpha/\beta(b-a) = 1$, would overlap the solid one. The dotted curve is with $a = 0.2$, $b = 0.3$ and has the other parameter values as the solid curve. The abscissa is frequency.

Figure 3. Imaginary part of the complex dielectric or anelastic parameter defined in formula (6) ; the interval of integration is defined by $a = 0.2$, $b = 0.8$. In all curves is assumed $b = f$, $c = a$, $\eta/\beta = 1$, $\gamma/(b-a)\eta = 0.02$, $\alpha/\beta(b-a) = 1$.The dashed curve is obtained from the classic formula (35) of Cole and Cole (1941) with parameters $\varepsilon_0 = \eta/\beta = 1$and $\varepsilon_\infty/\varepsilon_0 = \gamma/\eta = 0.02$, $\alpha/\beta = 1$, $a' = 0.5$. Note that $a' = (a+b)/2$. The abscissa is frequency.

Figure 4. Real part of the complex dielectric or anelastic parameter defined in formula (6) ; the interval of integration is defined by $a = 0.2$, $b = 0.8$. In all curves is assumed $b = f$, $c = a$, $\eta/\beta = 1$, $\gamma/(b-a)\eta = 0.02$, $\alpha/\beta(b-a) = 1$.The dashed curve is obtained from the classic formula (35) of Cole and Cole (1941) with parameters $\varepsilon_0 = \eta/\beta = 1$and $\varepsilon_\infty/\varepsilon_0 = \gamma/\eta = 0.02$, $\alpha/\beta = 1$, $a' = 0.5$. Note that $a' = (a+b)/2$. The abscissa is frequency.

Figure 5. Filtering properties of the memory in equation (6') and (16) and discussion of fast and slow diffusion. The abscissa is frequency in arbitrary units. The ordinate is in arbitrary units. The asymptotic filtering properties are obtained from equation (23'). The fast and slow diffusion are obtained for $f < b$ (diverging to an infinite value for $\omega \to \infty$) and $f > b$ (converging to zero for $\omega \to \infty$) respectively. When $b = f = 0$ and $\alpha = 0$ the curve is converging to zero for $\omega \to \infty$.

Figure 6. Path of integration in the complex plane used to obtain the $LT^{-1}$of $\Psi(s)$ and $\Omega(s)$ (formulae (7)), $K(s)$ (formula (10)) and $\sum(s)$ (formula (26)).

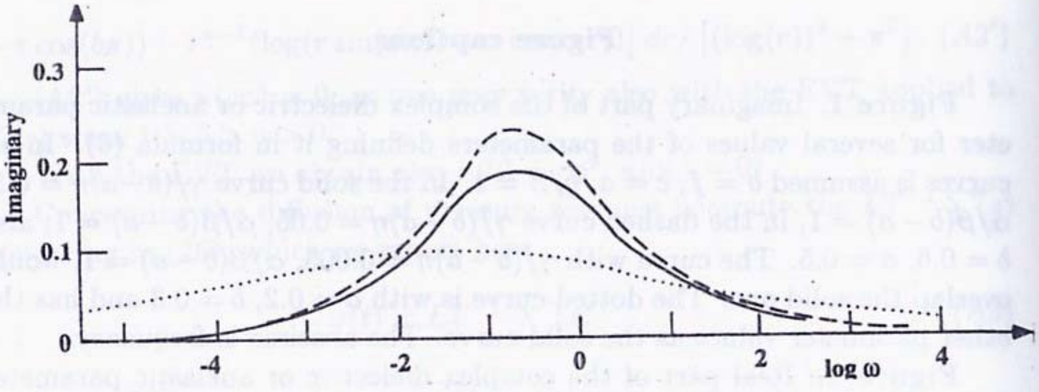


Figure 1

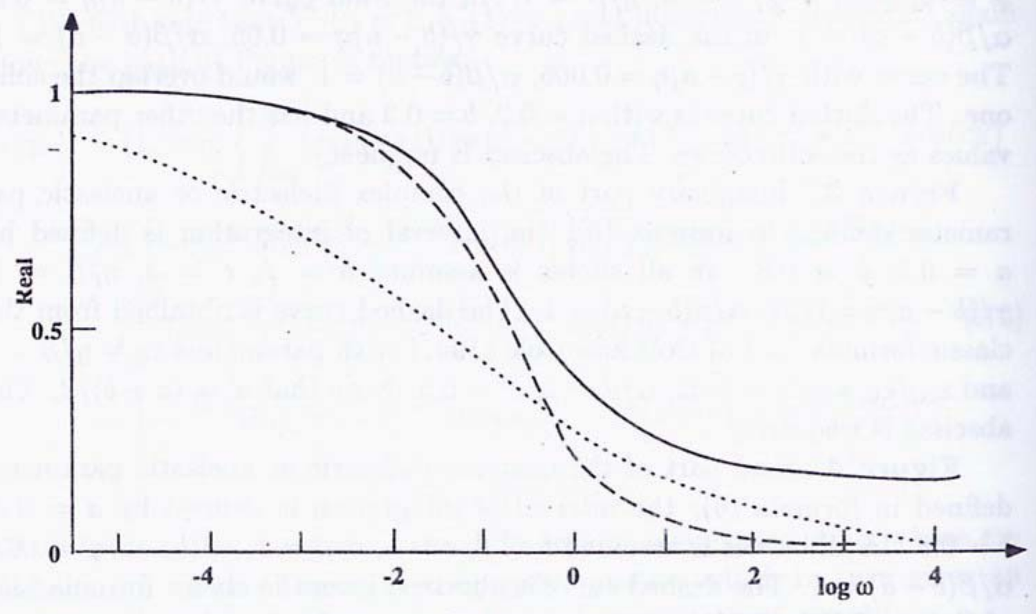


Figure 2

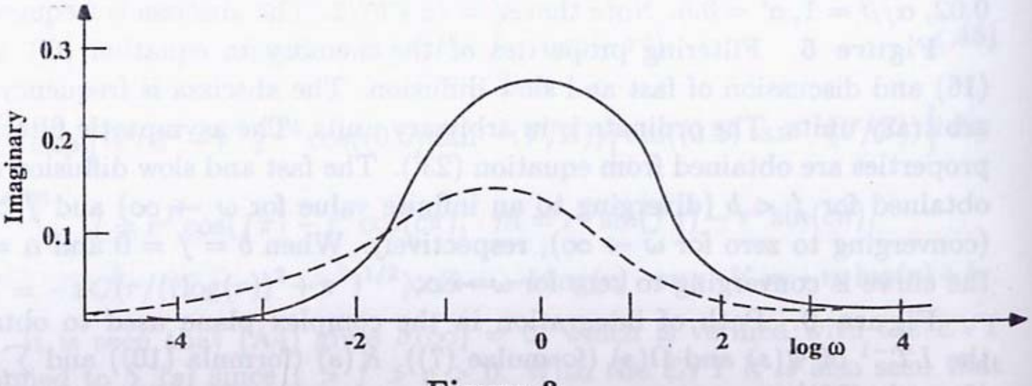


Figure 3

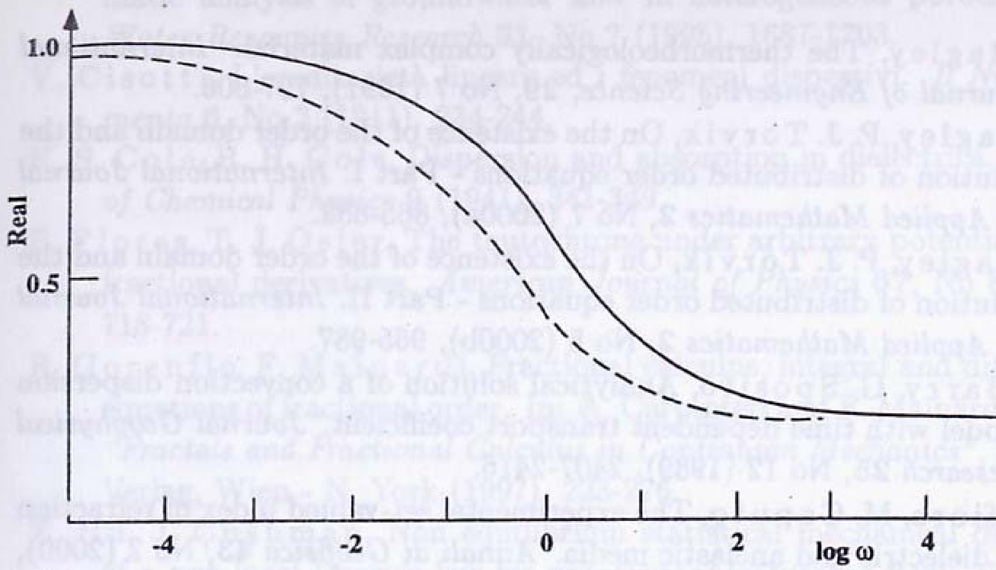
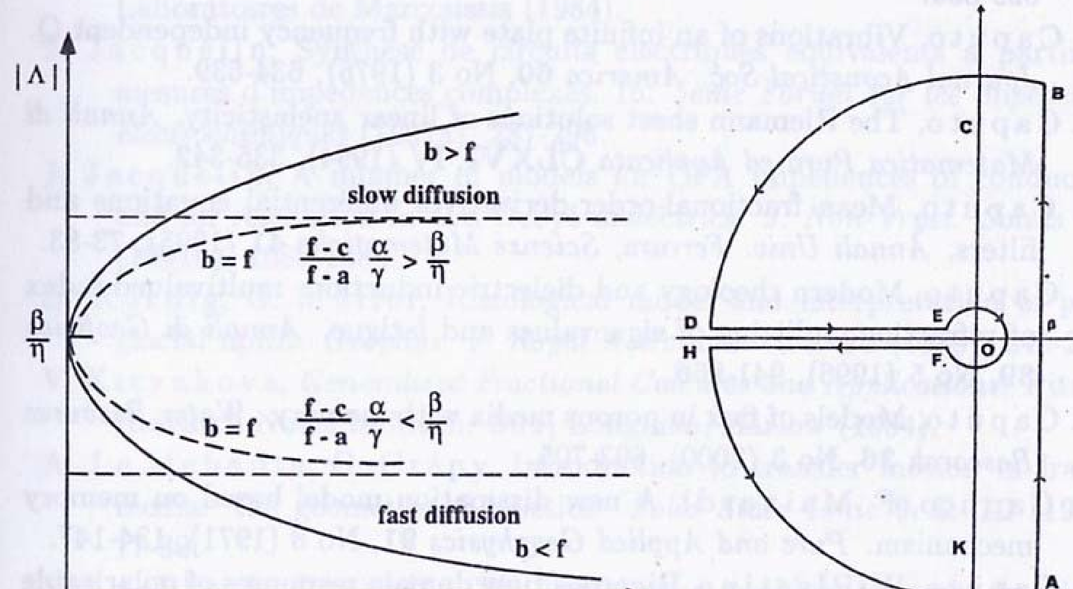


Figure 4



Figures 5, 6

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